Flow equations in the light-front QCD: mass gap and confinement

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The light-front QCD is studied using the method of flow equations. Solving the light-front gluon gap equation, the effective gluon mass is generated dynamically. The effective interaction between static quark and antiquark, generated through elimination of the quark-gluon minimal coupling by flow equations, has the Coulomb, $1/q^2$, and confining, $1/q^4$, singular behavior. Elimination of the quark-gluon coupling at small gluon momenta is governed by the cutoff dependent, dynamical gluon mass, which makes this elimination possible and provides such an enhancement at $q \sim 0$. The cutoff, which regulates small light-front x divergences, sets up a scale for the dynamical gluon mass and the string tension in the $q\bar{q}$-potential. The mechanism of confimenemet in the light-front frame is suggested, based on the singular nature of the light-front gauge along the light-front x-axis.

1 Introduction

The perturbative aspects of non-abelian gauge theories were underestood many years ago, and the perturbative calculations provided convincing proof that QCD is the theory of strong interactions. However nonperturbative QCD phenomena have been difficult to analyze mainly because calculational techniques are still lacking, even though the qualitative features have been more or less understood.

In particular, it is widely believed that pure Yang-Mills theory, with no dynamical quarks, posseses the features like asymptotic freedom, mass generation through the transmutation of dimensions and confinement: linear potential between static (probe) quarks. In the past few years there were several studies addressing the issue of confinement and generation of mass gap in the framework of the Schrödinger picture [1], [2]. To mention a few early works, see refs. [3]. We have tried to understand these nonperturbative mechanisms also in a Hamiltonian framework, solving flow equations for canonical QCD Hamiltonian in the light-front (LF) gauge selfconsistently for the few lowest sectors. Dynamics of quarks has been excluded to disentangle the complexity of chiral symmetry breaking. Early studies of confinement in the LF frame were conducted using similarity renormalization [4] and transverse lattice [5], based on the fact that QCD in $3+1$ already has a confining interaction in the form of the instantaneous four-fermion interaction, which is the confining interaction in

$1+1$. However, in our study the instantaneous interaction is canceled by the corresponding term in the dynamical interaction, generated by flow equations. The result is a three dimensional linear rising potential in the infrared region.

Our basic strategy has been to generate dynamically an effective gluon mass (energy) through interaction with the LF 'vacuum' (exact zero modes on the light-front, $x=0$), and construct an effective interaction between static quarks as an exchange with this dynamical gluon, which forms a flux between quarks. One could construct a LF field theory with massive photons and gluons, as was done in [6], but this theory is in a Coulomb phase rather than confining because of the lack of a dynamical mass generation mechanism. Dynamical mass generation in the light-front frame dates back to the works of Cornwall [7], where a kind of pinch technique was used to define a gluon propagator and hence a gluon effective energy, extracted from a Green's function for physical observables (for example two-body scattering amplitude). It was suggested in [7], that in physical terms, a gluon 'mass' may lead to a vortex condensation [a] with $< TrG_{\mu\nu}^2 >\neq 0$, and conversely $< TrG_{\mu\nu}^2 >\neq 0$ generates a gluon mass. This provides a link with the existing vortex picture of confinement.

A subtle point in the LF field theory is that the LF vacuum is just empty space. Therefore it seems a problem how confinement can occur in the LF frame, and what quantity sets up a scale for a dynamical mass gap and the string tension. We adopt a model, suggested by Susskind and Burkardt in the context of chiral symmetry breaking in the LF frame [8]. In the parton model one pictures a fast moving hadron as being some collection of constituents with relatively large momentum, such that when one boosts the system, doubles its momentum, all these partons double their momenta and so forth. Therefore one can formulate a field theory on the axis of the light-front momentum x. Partons that form a hadron are at positive, finite x and according to Feynman and Bjorken fill the x-axis in a way which gets denser and denser as one goes to smaller x; and the vacuum is at $x=0$. The fundamental property of LF Hamiltonians, that under a rescaling of the LF momentum, $x \to \lambda x$, the LF Hamiltonian scales like $H \to H/\lambda$, can be interpreted as a dilatation symmetry along the x-axis, if one thinks of the x-axis as a spacial axis. This symmetry holds on a classical level and it is brocken on a quantum level by anomali. As one approaches small x, interaction between partons gets stronger, contributing divergent matrix elements. A natural cutoff is provided by $\delta x = \varepsilon x$, a minimal spacing between constituents, which plays the role of UV-regulator [b]. As long as δx is finite, i.e. as long as the density of partons on the x-axis is not infinite,

[a] In the massive gauge theory one needs to introduce a scalar field, which condenses, in order to preserve gauge invariance. Topologically this field represents vortices [7].
[b] Small x correspond to the large light-front energies.

one obtains finite matrix elements. Cutoff δx breaks the dilatation symmetry along the x-axis and gauge symmetry, and sets up a scale for quantities of dimension of energy (mass) and higher powers in energy. In particular, one generates dynamically an effective gluon mass, which in turn will define string tension between quark and antiquark. Formation of the $q\bar{q}$ bound state through breaking an internal symmetry can be viewed analogously to the creation of Cooper pairs in superconducter.

In terms of effective theory, the Hamiltonian below the LF cutoff, $H_{\leq\varepsilon}$, which desribes high energy (UV) strongly correlated states, can be substituted by the v.e.v., since strongly coupled configurations are frozen. Hamiltonian above the cutoff $H_{>\varepsilon}$ is treated by flow equations (renormalization group transformation). One way of looking at the physics behind this v.e.v. and mass generation in gluodynamics is that the composite field ϕ which creates 0^+ glueballs has a finite v.e.v. [7],[2].

The dilatation symmetry, discussed above, reflects some underlying scale invariance of the LF field theory formulated on x-axis. Introducing the cutoff, breaks this symmetry. Because physics should remain independent on the cutoff, one must be looking for a fixed point of the renormalization group. Therefore the right tool for studing such a system is the renormalization group, which is provided by the method of flow equations [9].

In the method of flow equations an effective interaction between quarks arises through elimination of quark-gluon coupling. Procedure converges in the UV for large gluon momenta transfer $\vec{q}^{\,2}$, but in the IR one is not able to eliminate vanishing gluon momenta, because the similarity factor which governes the elimination does not decay for vanishing arguments. In physical terms, one can not integrate soft gluons in the same fashion. As soon as gluon aquires a dynamical mass, which vanishes at large gluon momenta and not equal zero only at small momenta, the similarity factor decays even for small momenta transfer, because the argument contains an effective gluon energy instead of only gluon mometum. Therefore we can eliminate by flow equations even soft gluon modes, that are responsible for the long-range part of an effective quark potential. In the IR this gives $1/q^4$ behavior for an effective $q\bar{q}$ interaction.

In the next section we solve flow equations for an effective gluon mass and quark-antiquark effective interaction, based on the QCD Hamiltonian in the light-front gauge.

2 Gluon mass gap and an effective quark interaction

Let Q being a projection operator on a one-gluon state, and P on a $q\bar{q}$ state, $Q|\psi\rangle = |g\rangle$ and $P|\psi\rangle = |q\bar{q}\rangle$. Flow equations for matrix elements of the QCD

Hamiltonian between these states read

$$\frac{dE_q(l)}{dl} = -\sum_p \frac{1}{E_q(l) - E_p(l)} \frac{d}{dl} \left(h_{qp}(l) h_{pq}(l) \right) \tag{1}$$

$$\frac{dh_{pp'}(l)}{dl} = -\sum_q \left(\frac{dh_{pq}(l)}{dl} \frac{1}{E_p(l) - E_q(l)} h_{qp'}(l) + h_{pq}(l) \frac{1}{E_{p'}(l) - E_q(l)} \frac{dh_{qp'}(l)}{dl} \right),$$

where p (p') runs through all states in the P-subspace, and q in the Q-subspace. Flow equations for the quark-gluon coupling h_{pq} (coupling between P and Q sectors) are

$$\begin{aligned} \frac{dh_{pq}(l)}{dl} &= -\left(E_p(l) - E_q(l) \right) \eta_{pq}(l) \\ \eta_{pq}(l) &= -\frac{h_{pq}(l)}{E_p(l) - E_q(l)} \frac{d}{dl} \left(\ln f(z_{pq}(l)) \right) \\ z_{pq}(l) &= l \left(E_p(l) - E_q(l) \right)^2 , \end{aligned} \tag{2}$$

where f is the similarity function with properties $f(0) = 1$ and $f(z \to \infty) = 0$. From Eq. (2), elimination of quark-gluon coupling

$$h_{pq}(l) = h_{pq}(0) f(z_{pq}(l)) , \tag{3}$$

is governed by the similarity function $f(l(E_p(l) - E_q(l))^2)$, which vanishes for the matrix elements with energy differences exceeding the cutoff λ, $|E_p(l) - E_q(l)| \gg 1/\sqrt{l} = \lambda$, $h_{pq}(0)$ is the initial value, and $E_q(l)$ ($E_p(l)$) is a solution of the flow (renormalization group) equation, Eq. (1). As long as the argument of the similarity function is not equal zero, one can eliminate quark-gluon coupling and solve flow equations for an effective gluon energy and effective $q\bar{q}$-interaction, Eq. (1). However, if one neglects the cutoff dependence of energies, $E_p(l) \sim E_p(0)$, the argument of the similarity function may become zero in the degenerate case, $E_p(0) = E_q(0)$, and then the energy denominator blows up in effective interactions in Eq. (1).

In physical terms, the leading behavior of the argument in the quark-gluon similarity function is given by an effective gluon energy transfer between quark and antiquark, $(E_p(l) - E_q(l)) \sim E^{eff}_{gluon}(l)$, which reduces at the initial value of flow parameter, $l = 0$, to a gluon momentum transfer, $E^{eff}_{gluon}(0) \sim \vec{q}^{\,2}$. Neglecting the cutoff dependence of the effective gluon energy, one eliminates quark-gluon coupling only at large gluon momenta, and fails at small gluon momenta, because the argument of the similarity function tends to zero. For small gluon momenta the dependence of gluon effective energy on the cutoff

becomes important, given by a dynamically generated gluon mass, that makes elimination of quark-gluon coupling possible even for small gluon momenta. We therefore solve flow equations, Eq. (1), consistently for an effective gluon mass and an effective $q\bar{q}$-interaction below. We show, that elimination of quark-gluon coupling for high and low gluon momenta contribute correspondingly to the UV and IR parts of quark-antiquark potential, that behave in momentum space like $1/\vec{q}^{\,2}$ and $1/\vec{q}^{\,4}$, respectively, where $\vec{q}$ is a gluon momentum transfer.

2.1 Gluon gap equation

In the light-front frame flow equation for an effective gluon mass (the first equation in Eq. (1)), with connection to the light-front energy $q^- = (q_\perp^2 + \mu^2(\lambda))/q^+$, reads

$$\begin{aligned}\frac{d\mu^2(\lambda)}{d\lambda} &= 2T_f N_f \int_0^1 \frac{dx}{x(1-x)} \int_0^\infty \frac{d^2k_\perp}{16\pi^3} g_q^2(\lambda) \frac{1}{Q_2^2(\lambda)} \frac{df^2(Q_2^2(\lambda);\lambda)}{d\lambda} \\ &\times \left(\frac{k_\perp^2 + m^2}{x(1-x)} - 2k_\perp^2 \right) \\ &+ 2C_a \int_0^1 \frac{dx}{x(1-x)} \int_0^\infty \frac{d^2k_\perp}{16\pi^3} g_g^2(\lambda) \frac{1}{Q_1^2(\lambda)} \frac{df^2(Q_1^2(\lambda);\lambda)}{d\lambda} \\ &\times \left(k_\perp^2 (1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}) \right) , \qquad (4)\end{aligned}$$

with

$$Q_1^2(\lambda) = \frac{k_\perp^2 + \mu^2(\lambda)}{x(1-x)} - \mu^2(\lambda) \ , \ Q_2^2(\lambda) = \frac{k_\perp^2 + m^2}{x(1-x)} - \mu^2(\lambda) \, , \qquad (5)$$

where gluon couples to the quark-anti-quark pairs, with the quark-gluon coupling $g_q(\lambda)$, and pairs of gluons, with the three-gluon coupling $g_g(\lambda)$; for finite λ these couplings are different from each other functions of momenta; connection between flow parameter and the cutoff, $l = 1/\lambda^2$, is used; similarity function f plays the role of UV regulator in loop integrals; current quark mass is m; for $SU(N_c)$ $T_f\delta_{ab} = \mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta_{ab}$ and the adjoint Casimir is $C_a\delta_{ab} = f^{acd}f^{bcd} = N_c\delta_{ab}$, N_c is the number of colors (i.e., $N_c = 3$).

Coupled system of equations for the cutoff dependent, effective gluon and quark masses was first derived in [11]. Here, neglecting the cutoff dependence of quark mass this system is reduced to Eq. (4).

Generally, it is difficult to solve Eq. (4). One of the reasons is that this equation contains (unknown) running coulings; therefore the cutoff dependence of couplings is neglected below. Second, initial condition for Eq. (4) is not

known. The following renormalization condition to define the running gluon mass $\mu(\lambda)$ through the 'physical' mass is assumed [11]: the effective Hamiltonian at the scale λ has bosonic eigenstates with eigenvalues of the form $q^- = (q_\perp^2 + \tilde{\mu}^2)/q^+$ [11], i.e.

$$\frac{q_\perp^2 + \tilde{\mu}^2}{q^+}\langle q', q\rangle = \frac{q_\perp^2 + \mu^2(\lambda)}{q^+}\langle q', q\rangle - \int^\lambda d\lambda' \sum_p (\eta_{q'p}(\lambda')h_{pq}(\lambda') - h_{q'p}(\lambda')\eta_{pq}(\lambda')) \ , \qquad (6)$$

where $\tilde{\mu}$ denotes the 'physical' gluon mass; the generator η_{pq}, given by Eq. (2), eliminates the quark-gluon (three-gluon) coupling h_{pq}; $|q\rangle$ denotes a single effective gluon state with momentum q^+ and $q_\perp$, $\langle q'|q\rangle = 16\pi^3 q^+ \delta^{(3)}(q'-q)$. Neglecting the cutoff dependence of the gluon mass on the r.h.s. of Eq. (4), the renormalization point is choisen at $q^2 = \tilde{\mu}^2$ (for in- and out-going and intermediate state gluons). Third, severe divergences arise at small light-front momenta x and must be regularized. For this purpose, as suggested by Zhang and Harindranath in [10], the minimum cutoff u for transverse momentum $k_\perp$ is introduced. Therefore, flow equation, Eq. (4), is integrated in the finite limits, $[u;\lambda]$. Explicitly, from the similarity function in gluon loop (the second term in Eq. (4)) one has $u \le \tilde{Q}_1^2 = (k_\perp^2 + \tilde{\mu}^2)/x(1-x) - \tilde{\mu}^2 \le \lambda$, that restricts the transverse momentum to $k_{\perp min} = (u^2 + \tilde{\mu}^2)x(1-x) - \tilde{\mu}^2 \le k_\perp \le k_{\perp max} = (\lambda^2 + \tilde{\mu}^2)x(1-x) - \tilde{\mu}^2$, and the light-front momentum to $\tilde{\mu}^2/(u^2+\tilde{\mu}^2) \le x \le 1 - \tilde{\mu}^2/(u^2+\tilde{\mu}^2)$. In the same way one finds restrictions on mometa in the quark-gluon loop (the first term in Eq. (4)). Integrating Eq. (4) with all constraints discussed above and assuming the condition of Eq. (6), gives [12]

$$\mu^2(\lambda) = \tilde{\mu}^2 + \delta\mu_{PT}^2(\lambda) + \delta\mu_{dyn}^2(\lambda, \tilde{\mu}, u) \, . \qquad (7)$$

where the perturbative term

$$\delta\mu_{PT}^2(\lambda) = \frac{g^2}{4\pi^2}\lambda^2 \left(C_a(\ln\frac{u^2}{\tilde{\mu}^2} - \frac{11}{12}) + T_f N_f \frac{1}{3} \right) \, . \qquad (8)$$

reproduces the result of the LF perturbation theory, with renormalization point $q^2 = 0$, [10]. When renormalization is performed in the second oder, the perturbative mass correction is absorbed by the mass counterterm, $m_{CT}^2 = -\delta\mu_{PT}^2(\lambda = \Lambda \to \infty)$. After the perturbative renormalization is completed, the dynamical mass

$$\mu_{dyn}^2(\lambda) = \tilde{\mu}^2 + \delta\mu_{dyn}^2(\lambda, \tilde{\mu}, u) = \tilde{\mu}^2 + \sigma(\tilde{\mu}, u)\ln\frac{\lambda^2}{\tilde{\mu}^2}$$

$$\sigma(\tilde{\mu},u) = -\frac{g^2}{4\pi^2}\tilde{\mu}^2\left(C_a(-\frac{u^2}{\tilde{\mu}^2}+\ln\frac{u^2}{\tilde{\mu}^2}-\frac{5}{12})+T_fN_f(\frac{1}{3}+\frac{m^2}{\tilde{\mu}^2})\right) , \qquad (9)$$

is left. Note, that the gluon mass μ_{dyn} is generated dynamically by flow equations. In the limit $\tilde{\mu}\to 0$, one has $\sigma = \lim_{\tilde{\mu}\to 0}\sigma(\tilde{\mu},u) = u^2g^2C_a/2\pi^2$, and, as shown below, σ plays the role of the string tension between quark and antiquark. Scale u is introduced by regulating the divergences at small LF momenta, $x\sim 0$, (small LF x regularization). The value u sets up a scale for the dynamical gluon mass and the string tension.

2.2 Effective $q\bar{q}$-interaction

An effective interaction, generated between quark and antiquark by flow equations in the LF frame (the second equation in Eq. (1)), reads

$$V_{q\bar{q}} = -Const\langle\gamma^\mu\gamma^\nu\rangle\lim_{\tilde{\mu}\to 0}B_{\mu\nu} , \qquad (10)$$

where instead of the strong coupling α_s some constant, $Const$, is introduced; $\langle\gamma^\mu\gamma^\nu\rangle$ is a current-current term in the exchange channel. At the end we set the gluon mass renormalization point ('physical' gluon mass) to zero, $\tilde{\mu}\to 0$; thus eliminating the sensitivity to the renormalization point. The kernel includes dynamical interaction and instantaneous exchange with nonzero dynamical gluon mass, given by

$$B_{\mu\nu} = g_{\mu\nu}\,(I_1+I_2)+\eta_\mu\eta_\nu\frac{\delta Q^2}{q^{+2}}\,(I_1-I_2) , \qquad (11)$$

where $g_{\mu\nu}$ is the LF metric tensor, and the LF vector η_μ is defined as $\eta\cdot k = k^+$. The cutoff dependence of four-momentum transfers along quark and antiquark lines is accumulated in the factor

$$I_1 = \int_0^\infty d\lambda\frac{1}{Q_1^2(\lambda)}\frac{df(Q_1^2(\lambda);\lambda)}{d\lambda}f(Q_2^2(\lambda);\lambda) , \qquad (12)$$

with mometum transfers defined in the light-front frame as

$$\begin{aligned} Q_1^2(\lambda) &= Q_1^2+\mu_{dyn}^2(\lambda)\ ,\ Q_1^2 = \frac{(x'k_\perp - xk'_\perp)^2+m^2(x-x')^2}{xx'} \\ Q_2^2(\lambda) &= Q_2^2+\mu_{dyn}^2(\lambda)\ ,\ Q_2^2 = Q_1^2|_{x\to(1-x);\ x'\to(1-x')} , \end{aligned} \qquad (13)$$

and the dynamical gluon mass μ_{dyn} is given by Eq. (9); also the following momenta are used $Q^2 = (Q_1^2+Q_2^2)/2$ and $\delta Q^2 = (Q_1^2-Q_2^2)/2$. Calculating

Eq. (10) with an explicit form of similarity function, gives in the leading order $\delta Q^2 \ll Q^2$ the following $q\bar{q}$-interaction [12]

$$V_{q\bar{q}} = -\langle \gamma^\mu \gamma_\mu \rangle \left(C_f \alpha_s \frac{4\pi}{Q^2} + \sigma \frac{8\pi}{Q^4} \right) , \tag{14}$$

where to this order Q^2 reduces in the instant frame to the square of gluon momentum transfer, $Q^2 \sim \vec{q}^{\,2}$, with the gluon momentum $\vec{q} = (q_z, q_\perp)$. In order to reproduce the standard Coulomb and linear confining potentials, the correct prefactors are restored, using the freedom to fit *Const* and σ terms. Confining term in Eq. (14), with singular behavior like $1/\vec{q}^{\,4}$, arise from the elimination of the quark-gluon coupling at small gluon momenta, that is governed by the cutoff dependent, dynamical gluon mass.

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